New bound and scattering state solutions of the Manning–Rosen potential with the centrifugal term

2009 J. Phys. A: Math. Theor. 42 205306

(http://iopscience.iop.org/1751-8121/42/20/205306)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 205306 (13pp) doi:10.1088/1751-8113/42/20/205306

New bound and scattering state solutions of the Manning–Rosen potential with the centrifugal term

Wen-Chao Qiang, Kai Li and Wen-Li Chen

Faculty of Science, Xi'an University of Architecture and Technology, Xi'an 710055, People's Republic of China

E-mail: qwcqj@pub.xaonline.com

Received 19 September 2008, in final form 4 March 2009
Published 5 May 2009
Online at stacks.iop.org/JPhysA/42/205306

Abstract

We proposed a new approximate scheme for a centrifugal term. Using new approximate formula for $1/r^2$, we obtained the bound state and scattering state solutions of the Manning–Rosen potential with centrifugal terms. All approximate analytical formulae of energy eigenvalues, normalized wavefunctions and scattering phase shifts are presented. In addition, we also suggested another much better approximate formula to $1/r^2$ for bound states. All data calculated by the above approximate analytical formulae are compared with those obtained by using the numerical integration method in the bound state and scattering state cases. Furthermore, the complete s-wave scattering state solutions for the Manning–Rosen potential are also naturally derived.

PACS numbers: 03.65.−w, 03.65.Ge, 03.65.NK

1. Introduction

It is well known that the Manning–Rosen potential is an important exponential-type potential [1, 2] which is given by

$$V(r) = \frac{1}{\kappa\beta^2}\left[\frac{\alpha(\alpha-1)\,\mathrm{e}^{-2r/\beta}}{(1-\mathrm{e}^{-r/\beta})^2} - \frac{A\,\mathrm{e}^{-r/\beta}}{1-\mathrm{e}^{-r/\beta}}\right], \qquad \kappa = 2\mu/\hbar^2, \tag{1}$$

where A and α are two dimensionless parameters [3], but parameter β has the dimension of length. This potential has been used as a model to study the energy eigenvalues of diatomic molecules and has important properties. It remains invariant by mapping $\alpha \leftrightarrow 1-\alpha$ and has a minimum value $V(r_0) = -\frac{A^2}{4\beta^2\alpha(\alpha-1)}$ at $r_0 = \beta\ln\left[1+\frac{2\alpha(\alpha-1)}{A}\right]$ for $\alpha > 1$. Furthermore, this potential reduces to the Hulthén potential [4] for $\alpha = 0$ or 1.

The Manning–Rosen potential has attracted much attention of researchers. Diaf *et al* investigated this potential for *s*-wave ($l = 0$) by the path integral approach [5]. Dong *et al* also obtained the bound state *s*-wave solutions of this potential [3]. We further found the bound state solutions of this potential with any l values [6]. Sameer *et al* obtained approximate l-state solutions of the Manning–Rosen potential by the Nikiforov–Uvarov (NU) method [7]. In addition, they found approximate l-state solutions of the D-dimensional Schrödinger equation for the same potential [8]. On the other hand, the study of the scattering state solutions of the Schrödinger equation with the Manning–Rosen potential is progressing. Chen *et al* presented exact solutions of the scattering states for the *s*-wave with this potential [9]. Wei *et al* found the approximately analytical scattering state solutions of the l-wave Schrödinger equation for the Manning–Rosen potential by a proper approximation to the centrifugal term [10].

Analyzing above-mentioned works, we see that in order to overcome the difficulty to solve the Schrödinger equation with the centrifugal term, all authors have used some approximation for $1/r^2$. It is obvious, in all cases of $l \neq 0$, whatever the bound state or the scattering state, that all solutions are approximate ones. In the case of the bound state, these approximations cannot always give results in good agreement with that obtained by the numerical integration method for any l values and some parameter values of the potential [6]. In the case of the scattering state, the data calculated by the approximate phase shift formula were never compared with any numerical results [10]. Therefore, one cannot know the veracity of the approximation used for the scattering state under consideration. This situation strongly suggests one should find a much better approximate expression for the centrifugal term to solve the Schrödinger equation with any l values for the bound and scattering states and judge its accuracy by comparing the approximate results with corresponding numerical data. This is just the aim of this paper. We shall develop a new approximation scheme which can be uniformly used to solve the Schrödinger equation with the centrifugal term for both bound state and scattering state.

This paper is organized as follows. In section 2 we derive an approximate expression for $1/r^2$ and the corresponding radial Schrödinger equation. Sections 3 and 4 are devoted to solving the bound states and scattering states, respectively, for the potential. Some numerical results and a more effective approximation to the centrifugal term only for bound states are presented in section 5. Some special cases of our results are discussed in the same section. The concluding remarks are given in section 6.

2. A new approximation scheme

The Schrödinger equation with natural units $\hbar = \mu = 1$ is given by

$$\left[-\tfrac{1}{2}\nabla^2 + V(r) - E\right]\psi(\mathbf{r}) = 0. \tag{2}$$

By taking $\psi(\mathbf{r}) = r^{-1}R(r)Y_{lm}(\theta, \phi)$ and considering potential (1), we obtain the radial Schrödinger equation as

$$\frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + \left[2E - \frac{1}{\beta^2}\left(\frac{\alpha(\alpha-1)\,\mathrm{e}^{-2r/\beta}}{(1-\mathrm{e}^{-r/\beta})^2} - \frac{A\,\mathrm{e}^{-r/\beta}}{1-\mathrm{e}^{-r/\beta}}\right) - \frac{l(l+1)}{r^2}\right]R(r) = 0, \tag{3}$$

which has no analytical solutions except for *s*-wave ($l = 0$) due to the centrifugal term. To find a quasi-analytical solution of this equation, we have to take some approximation for the centrifugal term. Instead of using the following approximate formula:

$$\frac{1}{r^2} \approx \frac{1}{\beta^2}\frac{\mathrm{e}^{-r/\beta}}{(1-\mathrm{e}^{-r/\beta})^2}, \tag{4}$$

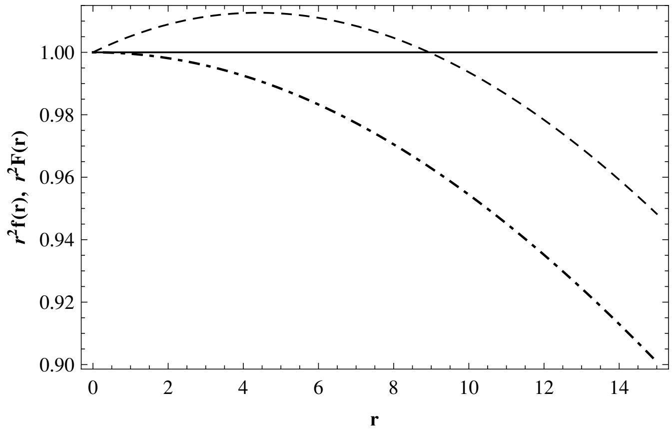

Figure 1. A graphic comparison of the variation of $r^2 f(r)$(dashed) and $r^2 F(r)$(dot-dashed) with r, where $f(r) = \frac{1}{\beta^2}\left[\frac{e^{1/\beta}\,e^{-r/\beta}}{1-e^{-r/\beta}} + \frac{e^{-2r/\beta}}{(1-e^{-r/\beta})^2}\right]$, $F(r) = \frac{1}{\beta^2}\frac{e^{-r/\beta}}{(1-e^{-r/\beta})^2}$, $\beta = 1/0.075$.

which has been used by many authors, here we propose another approximation scheme to $1/r^2$

$$\frac{1}{r^2} \approx \frac{1}{\beta^2}\left[\frac{e^{1/\beta}\,e^{-r/\beta}}{1-e^{-r/\beta}} + \frac{e^{-2r/\beta}}{(1-e^{-r/\beta})^2}\right]. \tag{5}$$

To illustrate the difference between the two approximation schemes, we plotted the variation of $\frac{r^2}{\beta^2}\left[\frac{e^{1/\beta}\,e^{-r/\beta}}{1-e^{-r/\beta}} + \frac{e^{-2r/\beta}}{(1-e^{-r/\beta})^2}\right]$ and $\frac{r^2}{\beta^2}\frac{e^{-r/\beta}}{(1-e^{-r/\beta})^2}$ for one typical value of parameter β in figure 1. It is obvious that for large β, $e^{1/\beta} \to 1$, the right-hand side of equation (5) approaches $\frac{e^{-r/\beta}}{\beta^2(1-e^{-r/\beta})^2}$, but it can greatly improve the behavior of the approximation to $1/r^2$ when β is small[1]. Substituting this equation into equation (3) and simplifying, we obtain

$$\frac{d^2R(r)}{dr^2} + \left[2E - \frac{1}{\beta^2}\left(\frac{\alpha'(\alpha'-1)\,e^{-2r/\beta}}{(1-e^{-r/\beta})^2} - \frac{A'\,e^{-r/\beta}}{1-e^{-r/\beta}}\right)\right]R(r) = 0, \tag{6}$$

where

$$\alpha' = \tfrac{1}{2}\left[1 + \sqrt{(2l+1)^2 + 4\alpha(\alpha-1)}\right], \qquad A' = A - l(l+1)\,e^{1/\beta}. \tag{7}$$

It is worth noting that α' is equal to $1+\delta$ in [6] and λ of [10]. Equation (6) is in the same form as equation (3) with $l = 0$ and can be solved analytically. Letting

$$z = e^{-r/\beta} \tag{8}$$

and substituting it into equation (6) leads to

$$z^2\frac{d^2R(z)}{dr^2} + z\frac{dR(z)}{dr} - \left[-2E\beta^2 - \frac{A'z}{1-z} + \frac{\alpha'(\alpha'-1)z^2}{(1-z)^2}\right]R(z) = 0. \tag{9}$$

[1] If we compute some Taylor series, it seems that the error in equation (4) is $O(1)$, as $r \to 0$, the error in (5) is $O(1/r)$. But in fact, if we let $x = 1/\beta$, and expand the right-hand side of equation (5) around $r = 0$ and $x = 0$, we would see that though there is a $1/r$ term, it is proportional to x^2, as a result, for large $\beta(x \to 0)$, this term $\to 0$.

It is worth noting even though equations (6) and (9) with some shifted parameters have their partners in [3, 6], some calculations in [3, 6] are not complete or need to be improved. Therefore, in the following two sections, we shall solve equation (6) for the bound state and scattering state, respectively, instead of directly borrowing the results of [3, 6].

3. Bound state solutions

For the bound state, considering $R(z) \to 0$ at two boundaries

$$z \to \begin{cases} 0, & \text{when} \quad r \to \infty, \\ 1, & \text{when} \quad r \to 0, \end{cases} \tag{10}$$

and energy E is negative, we take the following radial wavefunction of the form:

$$R(z) = (1-z)^{\alpha'} z^{\lambda} F(z), \tag{11}$$

where

$$\lambda = \beta\sqrt{-2E}. \tag{12}$$

Substitution of this trial solution into equation (9) leads to the following hypergeometric equation [11]:

$$(1-z)zF''(z) + [2\lambda + 1 - z(2\alpha' + 2\lambda + 1)]F'(z) + [A' - \alpha'(1+2\lambda)]F(z) = 0, \tag{13}$$

whose solution is nothing but the hypergeometric functions

$$F(z) = {}_2F_1(a, b; c; z), \tag{14}$$

where

$$\begin{aligned} a &= \alpha' + \lambda - \sqrt{\lambda^2 + \alpha'(\alpha' - 1) + A'}, \\ b &= \alpha' + \lambda + \sqrt{\lambda^2 + \alpha'(\alpha' - 1) + A'}, \\ c &= 1 + 2\lambda. \end{aligned} \tag{15}$$

By considering the finiteness of the solutions, the quantum condition is given by

$$\alpha' + \lambda - \sqrt{\lambda^2 + \alpha'(\alpha' - 1) + A'} = -n_r, \qquad n_r = 0, 1, 2, \ldots, [\sqrt{A - \alpha(\alpha - 1)} - \alpha'], \tag{16}$$

where $[f]$ denotes the largest integer inferior to f. From equation (16) we have

$$\lambda = -\frac{n_r^2 - A' + (1 + 2n_r)\alpha'}{2(n_r + \alpha')}. \tag{17}$$

Substitution of this equation into equation (12) yields the energy eigenvalues

$$\begin{aligned} E &= -\frac{1}{2\beta^2}\left[\frac{n_r^2 - A + \mathrm{e}^{\frac{1}{\beta}} l(l+1) + (2n_r + 1)\alpha'}{2(n_r + \alpha')}\right]^2, \\ n_r &= 0, 1, 2, \ldots [\sqrt{A - \alpha(\alpha - 1)} - \alpha'], \end{aligned} \tag{18}$$

which is slightly different from equation (15) of [6] because of the factor $\mathrm{e}^{1/\beta}$ before $l(l+1)$.

We now turn to the eigenfunction. Using equation (16) we can write the radial wavefunction as

$$R(z) = N(1-z)^{\alpha'} z^{\lambda} {}_2F_1(-n_r, n_r + 2(\alpha' + \lambda); 2\lambda + 1, z), \tag{19}$$

where N is a normalization constant to be determined from the normalization condition $\int_0^\infty R(r)^2 \, \mathrm{d}r = 1$. This normalization condition can be further written as

$$\beta N^2 \int_0^1 (1-z)^{2\alpha'} z^{2\lambda - 1} [{}_2F_1(-n_r, n_r + 2(\alpha' + \lambda), 2\lambda + 1, z)]^2 \, \mathrm{d}z = 1, \tag{20}$$

from which and by using the integral formula [12]

$$\begin{aligned} &\int_0^1 (1-z)^{2(\delta+1)} z^{2\lambda-1} {}_2F_1(-n, 2(\delta + \lambda + 1) + n; 2\lambda + 1; z)^2 \, \mathrm{d}z \\ &\quad = \frac{(n + \delta + 1) n! \Gamma(n + 2\delta + 2) \Gamma(2\lambda) \Gamma(2\lambda + 1)}{(n + \delta + \lambda + 1) \Gamma(n + 2\lambda + 1) \Gamma(2(\delta + \lambda + 1) + n)}, \qquad \delta > -\frac{3}{2} \bigwedge \lambda > 0, \end{aligned} \tag{21}$$

we obtain the analytical expression of the normalization constant

$$N = \frac{1}{\Gamma(2\lambda)} \sqrt{\frac{(n_r + \alpha' + \lambda) \Gamma(n_r + 2\lambda + 1) \Gamma(2(\alpha' + \lambda) + n_r)}{2\beta\lambda n_r! (n_r + \alpha') \Gamma(n_r + 2\alpha')}}. \tag{22}$$

This expression of the normalization constant, which can also be expressed by δ used in [6] instead of α', is more compact and concise than equation (18) in [6].

4. Scattering state solutions

We now turn to solve equation (9) for scattering states. For this purpose and the convenience of later calculation, we make the following variable change:

$$x = 1 - z \tag{23}$$

and define

$$k = \sqrt{2E}. \tag{24}$$

Then equation (9) becomes

$$\frac{\mathrm{d}^2 R(x)}{\mathrm{d}x^2} - \frac{1}{1-x} \frac{\mathrm{d}R(x)}{\mathrm{d}x} + \left(\frac{k^2 \beta^2}{(x-1)^2} - \frac{A'}{(x-1)x} - \frac{(\alpha' - 1)\alpha'}{x^2} \right) R(x) = 0. \tag{25}$$

Considering the boundary condition of the scattering states, we take the following trial wavefunction:

$$R(x) = x^{\alpha'} (1-x)^{-\mathrm{i}k\beta} f(x) \tag{26}$$

and substitute it into equation (25), thus yielding the following equation for $f(x)$:

$$(1-x)x \frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} + [2\alpha' + (2\mathrm{i}k\beta - 2\alpha' - 1)x] \frac{\mathrm{d}f(x)}{\mathrm{d}x} + [A' + \alpha'(2\mathrm{i}k\beta - 1)] f(x) = 0, \tag{27}$$

which is a hypergeometric equation, so its solution is a hypergeometric function

$$f(x) = {}_2F_1(a, b; c; x), \tag{28}$$

where

$$
\begin{aligned}
a &= \alpha' - \mathrm{i}k\beta - \sqrt{A' - k^2\beta^2 + \alpha'(\alpha' - 1)}, \\
b &= \alpha' - \mathrm{i}k\beta + \sqrt{A' - k^2\beta^2 + \alpha'(\alpha' - 1)}, \\
c &= 2\alpha'.
\end{aligned} \tag{29}
$$

From equations (26), (28), (29), we can write down the radial wavefunction of the scattering state as

$$
R(r) = C(1 - \mathrm{e}^{-r/\beta})^{\alpha'}\,\mathrm{e}^{\mathrm{i}kr}\,{}_2F_1(a, b; c; 1 - \mathrm{e}^{-r/\beta}), \tag{30}
$$

where C is a normalization constant. We now find the asymptotic expression of the above function for large r. For this purpose, using the following transformation formula of the hypergeometric function [11][2]

$$
\begin{aligned}
{}_2F_1(a, b; c; x) = {} & \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\,{}_2F_1(a, b; a + b - c + 1; 1 - x) \\
& + (1 - x)^{c-a-b}\frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c - a, c - b; c - a - b + 1; 1 - x)
\end{aligned} \tag{31}
$$

and ${}_2F_1(a, b; c; 0) = 1$, we obtain

$$
\begin{aligned}
{}_2F_1(a, b; c; 1 - \mathrm{e}^{-r/\beta}) = {} & \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\,{}_2F_1(a, b; a + b - c + 1; \mathrm{e}^{-r/\beta}) \\
& + (\mathrm{e}^{-r/\beta})^{c-a-b}\frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c - a, c - b; c - a - b + 1; \mathrm{e}^{-r/\beta}) \\
& \underset{r\to\infty}{\longrightarrow} \Gamma(c)\left[\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} + \mathrm{e}^{-2\mathrm{i}kr}\frac{\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}\right].
\end{aligned} \tag{32}
$$

From equation (29) it is easy to see $a + b - c = (c - a - b)^*$, $a = (c - b)^*$ and $b = (c - a)^*$, where x^* denotes the complex conjugate of x. So

$$
\frac{\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} = \left(\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\right)^*. \tag{33}
$$

Letting

$$
\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} = \left|\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\right|\mathrm{e}^{\mathrm{i}\delta} \tag{34}
$$

then

$$
\frac{\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} = \left|\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\right|\mathrm{e}^{-\mathrm{i}\delta}, \tag{35}
$$

where δ is a real number. Therefore, we get the asymptotic expression of ${}_2F_1(a, b; c; 1 - \mathrm{e}^{-r/\beta})$ for large r,

$$
{}_2F_1(a, b; c; 1 - \mathrm{e}^{-r/\beta}) \underset{r\to\infty}{\longrightarrow} \Gamma(c)\left|\frac{\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}\right|(\mathrm{e}^{\mathrm{i}\delta} + \mathrm{e}^{-\mathrm{i}(2kr+\delta)}). \tag{36}
$$

[2] http://functions.wolfram.com/07.23.17.0058.01.

Substituting this formula into equation (30), we finally obtain

$$R(r) \underset{r\to\infty}{\longrightarrow} C\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|(\mathrm{e}^{\mathrm{i}(kr+\delta)}+\mathrm{e}^{-\mathrm{i}(kr+\delta)})$$
$$= 2C\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\cos(kr+\delta)$$
$$= 2C\Gamma(c)\left|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\right|\sin\left(kr+\delta+\frac{\pi}{2}\right). \tag{37}$$

Comparing this formula with the general boundary condition of the scattering state wavefunction normalized on the '$k/2\pi$ scale' $R(r) = 2\sin\left(kr - \frac{\pi}{2}l + \delta_l\right)$, we obtain the phase shifts and the normalization constant

$$\delta_l = \frac{\pi}{2}(l+1) + \arg\Gamma(c-a-b) - \arg\Gamma(c-a) - \arg\Gamma(c-b)$$
$$= \frac{\pi}{2}(l+1) + \arg\Gamma(2\mathrm{i}k\beta)$$
$$- \arg\Gamma(\alpha' + \mathrm{i}k\beta + \sqrt{A + l(l+1)(1-\mathrm{e}^{1/\beta}) + \alpha(\alpha-1) - k^2\beta^2})$$
$$- \arg\Gamma(\alpha' + \mathrm{i}k\beta - \sqrt{A + l(l+1)(1-\mathrm{e}^{1/\beta}) + \alpha(\alpha-1) - k^2\beta^2}), \tag{38}$$

$$C = \frac{\left|\Gamma(\alpha' + \mathrm{i}k\beta + \sqrt{A + l(l+1)(1-\mathrm{e}^{1/\beta}) + \alpha(\alpha-1) - k^2\beta^2})\right|}{\Gamma(2\alpha')}$$
$$\times\left|\frac{\Gamma(\alpha' + \mathrm{i}k\beta - \sqrt{A + l(l+1)(1-\mathrm{e}^{1/\beta}) + \alpha(\alpha-1) - k^2\beta^2})}{\Gamma(2\mathrm{i}k\beta)}\right|. \tag{39}$$

It is well known that the poles of the scattering amplitude are corresponding to the bound states and the non-positive integers are the poles of the gamma function. So let

$$\alpha' + \mathrm{i}k\beta \pm \sqrt{A + l(l+1)(1-\mathrm{e}^{1/\beta}) + \alpha(\alpha-1) - k^2\beta^2} = -n_r, \qquad n_r = 0, 1, 2, \ldots, \tag{40}$$

we obtain

$$\lambda = \mathrm{i}\frac{-A + l(l+1)\,\mathrm{e}^{1/\beta} + n_r^2 + (2n_r+1) + \alpha'}{2\beta(n_r+\alpha')}. \tag{41}$$

Combining this equation with equation (12) yields the energy equation (18) of bound states again.

5. Numerical results and discussions

To show that the approximation scheme to $1/r^2$ is better than that in previous works for both bound state and scattering state, first, we tabulate the energy eigenvalues calculated by equation (18) (this work), equation (15) of [6] (previous) and MATHEMATICA package programmed by Lucha and Schöberl (Schroe) [13][3] respectively for the arbitrary principal quantum number n and the angular quantum number l with two different values of parameter α and some typical values of β in tables 1 and 2. Second, we tabulate the scattering phase shifts calculated by

[3] The numerical eigenvalues of energy E calculated by this package are reliable enough to be regarded as the exact values.

Table 1. Eigenvalues (18) as a function of β for 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = \mu = 1$) and for $\alpha = 0.75$, $A = 2\beta$.

States	$1/\beta$	This work	Another	Previous	Schroe
2p	0.025	−0.120 4190	−0.120 5270	−0.120 5793	−0.120 5271
	0.050	−0.107 8070	−0.108 2140	−0.108 4228	−0.108 2151
	0.075	−0.095 5883	−0.096 4433	−0.096 9120	−0.096 4469
3p	0.025	−0.045 8644	−0.045 8776	−0.045 9297	−0.045 8779
	0.050	−0.035 0357	−0.035 0589	−0.035 2672	−0.035 0633
	0.075	−0.025 5592	−0.025 5422	−0.026 0110	−0.025 5654
3d	0.025	−0.044 7380	−0.044 7737	−0.044 9299	−0.0447 743
	0.050	−0.033 6315	−0.033 6832	−0.034 3082	−0.033 6930
	0.075	−0.023 8084	−0.023 7106	−0.025 1168	−0.023 7621
4p	0.025	−0.020 8280	−0.020 8087	−0.020 8608	−0.020 8097
	0.050	−0.011 8288	−0.011 7209	−0.011 9292	−0.011 7365
	0.075	−0.005 3231	−0.005 0086	−0.005 4773	−0.005 0945
4d	0.025	−0.020 3587	−0.020 2993	−0.020 4555	−0.020 3017
	0.050	−0.011 2807	−0.0109 492	−0.0115 742	−0.010 9904
	0.075	−0.004 7632	−0.003 7985	−0.005 2047	−0.004 0331
4f	0.025	−0.020 0966	−0.019 9762	−0.020 2887	−0.019 9797
	0.050	−0.010 8506	−0.010 1784	−0.011 4284	−0.010 2393
	0.075	−0.004 2421	−0.002 2810	−0.005 0935	−0.002 6443
5p	0.025	−0.009 8396	−0.009 8055	−0.009 8576	−0.009 8079
5d	0.025	−0.009 6106	−0.009 5074	−0.009 6637	−0.009 5141
5f	0.025	−0.009 4783	−0.009 2712	−0.009 5837	−0.009 2825
5g	0.025	−0.009 3651	−0.009 0190	−0.009 5398	−0.009 0330
6p	0.025	−0.004 3951	−0.004 3531	−0.004 4051	−0.004 3583
6d	0.025	−0.004 2767	−0.004 1499	−0.004 3061	−0.004 1650
6f	0.025	−0.004 2067	−0.003 9528	−0.004 2652	−0.003 9803
6g	0.025	−0.004 1459	−0.003 7220	−0.004 2428	−0.003 7611

equation (38) (this work), by equation (19) of [10] (Wei's)[4] and by the amplitude-shift method (APM) [16–19] respectively for some values of k, which is equivalent to energy E, and a few angular quantum number l with two different values of parameter α and some typical values of β in tables 3–5. The data in all tables show that the approximate expression in equation (5) is globally better than $1/r^2 \approx \mathrm{e}^{-r/\beta}/(\beta^2(1-\mathrm{e}^{-r/\beta})^2)$ of [6] and [10] for both cases of bound states and scattering states.

It is notable that there are columns named as 'another' in tables 1 and 2 which need to be explained further. For this purpose, let

$$\frac{1}{r^2} \approx \left[c_0 + c_1\frac{\mathrm{e}^{-r/\beta}}{1-\mathrm{e}^{-r/\beta}} + c_2\frac{\mathrm{e}^{-2r/\beta}}{(1-\mathrm{e}^{-r/\beta})^2}\right], \tag{42}$$

and expand the right of above equation around $r = 0$ up to the first-order degree of r, we obtain

$$\frac{1}{r^2} \approx \frac{c_2\beta^2}{r^2} + \frac{(c_1-c_2)\beta}{r} + \left(c_0 - \frac{c_1}{2} + \frac{5c_2}{12}\right) + \frac{(c_1-c_2)}{12\beta}r, \tag{43}$$

[4] The general asymptotic expression of the scattering state should be proportional to $\sin(kr - \frac{1}{2}l\pi + \delta_l)$ [14, 15] instead of $\sin(kr + \delta_l)$ used by Wei [10]. We have added $\frac{1}{2}l\pi$ to the formula of the phase shift in Ref. [10] when we calculate.

Table 2. Eigenvalues (18) as a function of β for 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = \mu = 1$) and for $\alpha = 1.5$, $A = 2\beta$.

States	$1/\beta$	This work	Another	Previous	Schroe
2p	0.025	−0.089 9026	−0.089 9708	−0.090 0229	−0.089 9708
	0.050	−0.079 7878	−0.080 0389	−0.080 2472	−0.080 0400
	0.075	−0.070 0503	−0.070 5645	−0.071 0332	−0.070 5701
3p	0.025	−0.036 9119	−0.036 9130	−0.036 9651	−0.036 9134
	0.050	−0.027 2863	−0.027 2636	−0.027 4719	−0.027 2696
	0.075	−0.019 0308	−0.018 9163	−0.019 3850	−0.018 9474
3d	0.025	−0.039 4647	−0.039 4782	−0.039 6345	−0.039 4789
	0.050	−0.029 4664	−0.029 4379	−0.030 0629	−0.029 4496
	0.075	−0.020 6641	−0.020 4058	−0.021 8121	−0.020 4663
4p	0.025	−0.017 1972	−0.017 1728	−0.017 2249	−0.017 1740
	0.050	−0.009 0203	−0.008 8935	−0.009 1019	−0.008 9134
	0.075	−0.003 4325	−0.003 0791	−0.003 5478	−0.003 1884
4d	0.025	−0.018 2772	−0.018 2087	−0.018 3649	−0.018 2115
	0.050	−0.009 8329	−0.009 4697	−0.010 0947	−0.009 5167
	0.075	−0.003 8986	−0.002 8746	−0.004 2808	−0.003 1399
4f	0.025	−0.018 7428	−0.018 6098	−0.018 9223	−0.018 6137
	0.050	−0.010 0472	−0.009 3353	−0.010 5852	−0.009 4015
	0.075	−0.003 8657	−0.001 8402	−0.004 6527	−0.002 2307
5p	0.025	−0.008 1154	−0.008 0787	−0.008 1308	−0.008 0816
5d	0.025	−0.008 6417	−0.008 5340	−0.008 6902	−0.008 5415
5f	0.025	−0.008 8629	−0.008 6497	−0.008 9622	−0.008 6619
5g	0.025	−0.008 9536	−0.008 6002	−0.009 1210	−0.008 6150
6p	0.025	−0.003 5249	−0.003 4813	−0.003 5334	−0.003 4876
6d	0.025	−0.003 7940	−0.003 6647	−0.003 8209	−0.003 6813
6f	0.025	−0.003 9055	−0.003 6481	−0.003 9606	−0.003 6774
6g	0.025	−0.003 9492	−0.003 5214	−0.004 0422	−0.003 5623

which determines the expanding coefficients c_0, c_1 and c_2

$$c_0 = \frac{1}{12\beta^2}, \qquad c_1 = c_2 = \frac{1}{\beta^2}. \tag{44}$$

Equations (42) and (44) can be combined as

$$\frac{1}{r^2} \approx \frac{1}{\beta^2}\left[\frac{1}{12} + \frac{\mathrm{e}^{-r/\beta}}{(1-\mathrm{e}^{-r/\beta})^2}\right]. \tag{45}$$

This approximate expression for $1/r^2$ is equivalent to $\frac{1}{r^2} \approx \frac{\mathrm{e}^{-r/\beta}}{\beta^2(1-\mathrm{e}^{-r/\beta})^2}$ used by [6, 10] plus a constant $\frac{1}{12\beta^2}$. Substituting equation (45) into equation (3) and solving it for bound states, we obtain the energy eigenvalues which adds an energy-modifying term $\frac{l(l+1)}{24\beta^2}$ to equation (15) of [6]. Or using the present symbol α' we explicitly express this new energy eigenvalue formula as

$$E = \frac{1}{2\beta^2}\left\{\frac{l(l+1)}{12} - \left[\frac{n_r^2 - A + l(1+l) + (2n_r+1)\alpha'}{2(n_r+\alpha')}\right]^2\right\}, \tag{46}$$

where α' is defined by equation (7). This new energy formula gives the data in column 'another' in tables 1 and 2, which are more close to that calculated by MATHEMATICA

Table 3. Scattering phase shifts (38) as a function of $k = \sqrt{2E}$ and β in atomic units ($\hbar = \mu = 1$) for $\alpha = 0.75$, $\alpha = 1.5$, $A = 2\beta$ and $l = 1$.

		$\alpha = 0.75$			$\alpha = 1.50$		
k	$1/\beta$	This work	Wei's	APM	This work	Wei's	APM
1	0.025	4.078 377	4.080 845	4.068 534	3.498 435	3.500 815	3.488 585
	0.050	3.411 426	3.419 780	3.395 210	2.865 379	2.873 369	2.848 778
	0.075	3.025 185	3.042 095	3.005 469	2.507 983	2.524 046	2.487432
3	0.025	1.798 463	1.799 536	1.795 341	1.295 082	1.296 119	1.291 952
	0.050	1.567 100	1.570 864	1.562 570	1.079 695	1.083 308	1.075 098
	0.075	1.430 458	1.438 260	1.425 815	0.957 119	0.964 578	0.952 118
5	0.025	1.221 311	1.222 021	1.219 492	0.734 300	0.734 987	0.732 582
	0.050	1.081 530	1.084 051	1.079 156	0.605 317	0.607 747	0.602 703
	0.075	0.998 536	1.003 812	0.996 350	0.531 958	0.537 024	0.529 513
7	0.025	0.948 985	0.949 522	0.947 793	0.468 828	0.469 349	0.467 551
	0.050	0.848 817	0.850 741	0.847 193	0.376 938	0.378 796	0.375 245
	0.075	0.789 165	0.793 213	0.787 848	0.324 730	0.328 627	0.323 265
9	0.025	0.788 239	0.788 675	0.787 290	0.311 802	0.312 225	0.310 820
	0.050	0.710 170	0.711 738	0.708 987	0.240 500	0.242 016	0.239 238
	0.075	0.663 582	0.666 893	0.662 713	0.200 027	0.203 221	0.199050
11	0.025	0.681 352	0.681 720	0.680 576	0.207 235	0.207 593	0.206 441
	0.050	0.617 381	0.618 711	0.616 427	0.149 014	0.150 302	0.148 061
	0.075	0.579 146	0.581 961	0.578 578	0.115 996	0.118 715	0.115 352
13	0.025	0.604 773	0.605 093	0.604 086	0.132 235	0.132 546	0.131 600
	0.050	0.550 580	0.551 738	0.549 771	0.083 058	0.084 180	0.082227
	0.075	0.518 147	0.520 604	0.517 611	0.055 189	0.057 565	0.054712
15	0.025	0.547 024	0.547 307	0.546 498	0.075 625	0.075 900	0.075 055
	0.050	0.500 010	0.501 038	0.499 365	0.033 070	0.034 068	0.032230
	0.075	0.471 843	0.474 028	0.471 500	0.008 971	0.011 086	0.008 611

package 'Schroe' than that calculated by equation (18). This fact shows that equation (45) is a much better approximation for $1/r^2$ than equation (5) in the case of the bound state. But unfortunately, this approximation is unavailable to the scattering states[5].

We are now going to study the special case of our results. We focus our discussion on the s-wave case ($l = 0$). It is worth pointing out that when $l = 0, \alpha' = \alpha$, for $\alpha \geqslant 1/2$ and $\alpha' = 1 - \alpha$, for $\alpha < 1/2$. Therefore, from equations (18) and (7) we obtain the energy eigenvalues of bound states with $l = 0$

$$E = \begin{cases} -\dfrac{1}{2\beta^2}\left[\dfrac{A-\alpha}{2(n_r+\alpha)} - \dfrac{n_r(n_r+2\alpha)}{2(n_r+\alpha)}\right]^2, & \alpha \geqslant \dfrac{1}{2}; \\ -\dfrac{1}{2\beta^2}\left[\dfrac{A-(n_r+1)^2+(2n_r+1)\alpha}{2(n_r-\alpha+1)}\right]^2, & \alpha < \dfrac{1}{2}. \end{cases} \tag{47}$$

[5] If we use equation (45) to study the scattering states, calculations similar to that in section 4 lead us to compare $R(r) \underset{r\to\infty}{\longrightarrow} = 2C\Gamma(c)|\frac{\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}|\sin(k'r+\delta+\frac{\pi}{2})$ with $R(r) = 2\sin(kr - \frac{\pi}{2}l + \delta_l)$, where $k' = \sqrt{2(E - c_0 l(l+1))}$ and $k = \sqrt{2E}$. It is obvious that $k'r$ cannot completely cancel kr unless we take $k' \approx k$ in this case. This situation shows that equation (45) is not proper for the scattering state.

Table 4. Scattering phase shifts (38) as a function of $k = \sqrt{2E}$ and β in atomic units ($\hbar = \mu = 1$) for $\alpha = 0.75$, $\alpha = 1.5$, $A = 2\beta$ and $l = 2$.

		$\alpha = 0.75$			$\alpha = 1.50$		
k	$1/\beta$	This work	Wei's	APM	This work	Wei's	APM
1	0.025	3.567 518	3.574 115	3.536 993	3.256 749	3.263 224	3.226 085
	0.050	2.912 734	2.934 458	2.860 610	2.632 455	2.653 679	2.579780
	0.075	2.535 631	2.578 602	2.468 167	2.280 725	2.322 551	2.212 170
3	0.025	1.590 218	1.593 113	1.580 589	1.297 994	1.300 845	1.288454
	0.050	1.363 000	1.372 973	1.348 109	1.085 138	1.094 930	1.069 957
	0.075	1.228 530	1.248 921	1.211 534	0.963 128	0.983 110	0.945802
5	0.025	1.079 300	1.081 229	1.073 690	0.789 801	0.791 705	0.784200
	0.050	0.941 699	0.948 460	0.933 426	0.662 002	0.668 654	0.653 695
	0.075	0.859 432	0.873 425	0.850 910	0.588 379	0.602 124	0.579 602
7	0.025	0.835 674	0.837 144	0.831 736	0.547 074	0.548 525	0.543 099
	0.050	0.736 897	0.742 093	0.731 392	0.455 857	0.460 975	0.450 388
	0.075	0.677 469	0.688 298	0.672 271	0.403 158	0.413 810	0.397774
9	0.025	0.690 936	0.692 133	0.687 966	0.402 727	0.403 909	0.399 691
	0.050	0.613 847	0.618 103	0.609 798	0.331 844	0.336 039	0.327756
	0.075	0.567 256	0.576 166	0.563 785	0.290 812	0.299 583	0.287 143
11	0.025	0.594 256	0.595 269	0.592 031	0.306 243	0.307 245	0.303806
	0.050	0.531 019	0.534 641	0.527 866	0.248 297	0.251 869	0.245 128
	0.075	0.492 664	0.500 272	0.490 115	0.214 702	0.222 197	0.212034
13	0.025	0.524 752	0.525 635	0.522 908	0.236 846	0.237 718	0.234 879
	0.050	0.471 132	0.474 295	0.468 683	0.187 852	0.190 973	0.185 279
	0.075	0.438 515	0.445 177	0.436 533	0.159 413	0.165 979	0.157 268
15	0.025	0.472 195	0.472 978	0.470 441	0.184 348	0.185 121	0.182 624
	0.050	0.425 642	0.428 456	0.423 507	0.141 916	0.144 694	0.139667
	0.075	0.397 254	0.403 194	0.395 742	0.117 260	0.123 117	0.115637

This equation is the same as equation (21) of [6]. On the other hand, for the scattering states, from equations (38) and (39) we have phase shift and the normalization constant

$$\delta_l = \begin{cases} \dfrac{\pi}{2} + \arg\Gamma(2\mathrm{i}k\beta) \\ -\arg\Gamma(\alpha + \mathrm{i}k\beta + \sqrt{A + \alpha(\alpha-1) - k^2\beta^2}) \\ -\arg\Gamma(\alpha + \mathrm{i}k\beta - \sqrt{A + \alpha(\alpha-1) - k^2\beta^2}), & \alpha \geqslant \dfrac{1}{2}; \\ \dfrac{\pi}{2} + \arg\Gamma(2\mathrm{i}k\beta) \\ -\arg\Gamma(1 - \alpha + \mathrm{i}k\beta + \sqrt{A + \alpha(\alpha-1) - k^2\beta^2}) \\ -\arg\Gamma(1 - \alpha + \mathrm{i}k\beta - \sqrt{A + \alpha(\alpha-1) - k^2\beta^2}), & \alpha < \dfrac{1}{2}. \end{cases} \tag{48}$$

Table 5. Scattering phase shifts (38) as a function of $k = \sqrt{2E}$ and β in atomic units ($\hbar = \mu = 1$) for $\alpha = 0.75$, $\alpha = 1.5$, $A = 2\beta$ and $l = 3$.

		$\alpha = 0.75$			$\alpha = 1.50$		
k	$1/\beta$	This work	Wei's	APM	This work	Wei's	APM
1	0.025	3.251 157	3.263 163	3.188 826	3.049 200	3.061 072	2.986 780
	0.050	2.620 476	2.659 014	2.510 919	2.445 868	2.483 864	2.335816
	0.075	2.264 790	2.339 330	2.117 855	2.112 295	2.185 621	1.964 331
3	0.025	1.470 162	1.475 527	1.450 462	1.268 193	1.273 510	1.248 460
	0.050	1.249 831	1.268 042	1.218 083	1.061 025	1.079 047	1.029 207
	0.075	1.119 852	1.156 673	1.081 857	0.942 306	0.978 700	0.903 861
5	0.025	1.000 100	1.003 703	0.988 751	0.796 884	0.800 460	0.785 527
	0.050	0.866 094	0.878 570	0.848 637	0.671 952	0.684 314	0.654 320
	0.075	0.785 575	0.811 177	0.766 094	0.599 351	0.624 694	0.579 642
7	0.025	0.774 097	0.776 853	0.766 189	0.570 064	0.572 800	0.562 075
	0.050	0.677 620	0.687 265	0.665 798	0.480 625	0.490 189	0.468 856
	0.075	0.618 980	0.638 934	0.607 054	0.428 196	0.447 965	0.415 884
9	0.025	0.639 178	0.641 428	0.633 056	0.434 581	0.436 815	0.428 416
	0.050	0.563 720	0.571 649	0.555 048	0.364 923	0.372 789	0.356 121
	0.075	0.517 472	0.533 964	0.508 886	0.323 829	0.340 178	0.315 312
11	0.025	0.548 759	0.550 668	0.543 770	0.343 750	0.345 647	0.338 815
	0.050	0.486 752	0.493 519	0.479 904	0.286 701	0.293 417	0.279 710
	0.075	0.448 500	0.462 630	0.442 201	0.252 879	0.266 891	0.246 629
13	0.025	0.483 596	0.485 260	0.479 453	0.278 271	0.279 925	0.274 157
	0.050	0.430 942	0.436 865	0.425 394	0.229 963	0.235 843	0.224 334
	0.075	0.398 286	0.410 692	0.393 407	0.201 206	0.213 511	0.196 257
15	0.025	0.434 225	0.435 704	0.430 689	0.228 650	0.230 120	0.225 077
	0.050	0.388 454	0.393 735	0.383 699	0.186 758	0.192 000	0.181 955
	0.075	0.359 940	0.371 025	0.355 967	0.161 733	0.172 731	0.157 720

$$C = \begin{cases} \dfrac{|\Gamma(\alpha + \mathrm{i}k\beta + \sqrt{A + \alpha(\alpha-1) - k^2\beta^2})\Gamma(\alpha + \mathrm{i}k\beta - \sqrt{A + \alpha(\alpha-1) - k^2\beta^2})|}{\Gamma(2\alpha)|\Gamma(2\mathrm{i}k\beta)|}, \\ \qquad \alpha \geqslant \dfrac{1}{2}; \\ \dfrac{|\Gamma(1 - \alpha + \mathrm{i}k\beta + \sqrt{A + \alpha(\alpha-1) - k^2\beta^2})\Gamma(1 - \alpha + \mathrm{i}k\beta - \sqrt{A + \alpha(\alpha-1) - k^2\beta^2})|}{\Gamma(2(1-\alpha))|\Gamma(2\mathrm{i}k\beta)|}, \\ \qquad \alpha < \dfrac{1}{2}. \end{cases} \tag{49}$$

The first formulae of equations (48),(49) have been given without distinction between $\alpha \geqslant \frac{1}{2}$ and $\alpha < \frac{1}{2}$ in [9, 10]. However, the second formulae of the above two equations have been missed by authors of [9, 10]. We have calculated the phase shifts for both cases of $\alpha \geqslant \frac{1}{2}$ and $\alpha < \frac{1}{2}$ by APM and find that the first formula of equation (48) is only applicable to the case of $\alpha \geqslant \frac{1}{2}$. Nevertheless, the phase shifts calculated by the first formula of equation (48) do not coincide with that obtained by APM for $\alpha < \frac{1}{2}$, but those given by the second formulae of equation (48) do.

6. Concluding remarks

We have proposed a new approximation scheme for the centrifugal term with which we have obtained new approximate analytical solutions for the bound and scattering states with any l-state. For the bound state, the energy eigenvalues are given by equation (18), and the normalized wavefunctions are expressed by equations (19) and (22). For the scattering state, the phase shifts and wavefunctions normalized on the '$k/2\pi$ scale' are given by equations (30), (38) and (39). On the other hand, we have also numerically solved the Schrödinger equation with the Manning–Rosen potential as well as any l values for both bound state and scattering state. The comparison of numerical results with approximate ones in both bound and scattering state cases shows that our new approximate formula to $1/r^2$ is better than that used in the literature. Furthermore, we have developed another approximate formula for $1/r^2$ available to bound states. The energy eigenvalues calculated according to this formula, equation (46), are in better agreement with that obtained by the numerical integration method. Finally, from our results, we naturally derived the complete s-wave scattering state solutions for the Manning–Rosen potential. We hope that the results obtained in this paper could enlarge and enhance the application of the Manning–Rosen potential in the relevant fields of physics and the method used in this work could be used in other bound and scattering state problems.

Acknowledgments

The authors would like to thank the kind referees for the positive and invaluable suggestions which have improved the manuscript greatly. This work is supported by the Natural Science Basic Research Plan in Shaanxi Province of China (program no: SJ08A13).

References

[1] Manning M F and Rosen N 1933 *Phys. Rev.* **44** 953
[2] Rosen N and Morse P M 1932 *Phys. Rev.* **42** 210
[3] Dong S H and García-Ravelo J 2007 *Phys. Scr.* **75** 307
[4] Hulthén L 1942 *Ark. Mat. Astron. Fys.* A **28** 5
[5] Diaf A, Chouchaoui A and Lombard R J 2005 *Ann. Phys.* **317** 354
[6] Qiang W C and Dong S H 2007 *Phys. Lett.* A **368** 13
[7] Ikhdair S M and Sever R arXiv:0801.4271v1 [quant-ph]
[8] Ikhdair S M and Sever R arXiv:0801.3518v1 [quant-ph]
[9] Chen C Y, Lu F L and Sun D S 2007 *Phys. Scr.* **76** 428
[10] Wei G F, Long C Y and Dong S H 2008 *Phys. Lett.* A **372** 2592
[11] Gradshteyn I S and Ryzhik I M 1994 *Tables of Integrals, Series, and Products* 5th edn (New York: Academic)
[12] Abbott P 2007 Private communication
[13] Lucha W and Schöberl F F 1999 *Int. J. Mod. Phys.* C **10** 607
[14] Schiff L I 1955 *Quantum Mechanics* 3rd edn (New York: McGraw-Hill)
[15] Landau L D and Lifshitz E M 1977 *Quantum Mechanics, Non-Relativistic Theory* 3rd edn (Oxford: Pergamon)
[16] Milne W E 1930 *Phys. Rev.* **35** 863
[17] Wheeler J A 1937 *Phys. Rev.* **52** 1123
[18] Thylwe K E 2008 *J. Phys. A: Math. Theor.* **41** 115304
[19] Thylwe K E 2008 *Phys. Scr.* **77** 065005